On the Lie-Riemann-Helmholtz-Hilbert Problem of the Foundations of Geometry.

By Robert L. Moore.

§ 1. *Introduction.*

Concerning Hilbert's paper, "Über die Grundlagen der Geometrie,"* Poincaré says, according to Halsted's translation,† "As regards the ideas of Lie, the progress made is considerable. Lie supposed his groups defined by analytic equations. Hilbert's hypotheses are far more general. Without doubt this is still not entirely satisfactory, since though the *form* of the group is supposed any whatever, its *matter*, that is to say the plane which undergoes the transformations, is still subjected to being a *number-manifold* in Lie's sense. Nevertheless, this is a step in advance, and besides Hilbert analyzes better than anyone before him the idea of *number-manifold* and gives outlines which may become the germ of an assumptional theory of analysis situs."

The present paper contains a set of assumptions Σ in terms of the notions *point*, *region*, and *motion*. Here the space which undergoes the transformations (motions) is *not subjected in advance* to the condition of being a number plane nor is it *presupposed* that the *regions* are in one-to-one correspondence with portions of such a plane. Of course Poincaré's statement that "the form of the group is supposed any whatever" applies to its *presupposed* form. Hilbert's axioms so condition the form of the group in question as to necessitate that it should be simply isomorphic with the group of rigid motions in a space of two dimensions. It is largely, or entirely, a question of analysis. It may be said that Hilbert *analyzes* the group of transformations (motions) but leaves largely unanalyzed the space that undergoes the transformations. In the present treatment the "form" of the transformations and their "matter" (the space that is transformed by them) are subjected to what might be termed a *simultaneous analysis*.

* *Matematische Annalen*, Vol. LVI (1902–03), pp. 381–422.
† "The Bolyai Prize," *Science*, May 19, 1911, p. 765.

§ 2. *Preliminary Explanations and Definitions.*

I consider a class $\overline{S}$ of undefined elements called *points*, an undefined class of sub-classes of $\overline{S}$ called *regions* and an undefined class of one-to-one transformations of $\overline{S}$ into itself called *motions*.* If P is a point of $\overline{S}$, and M is a motion; the point into which P is transformed by M will be denoted by the symbol $M(P)$. If K is a point-set and M is a motion, $M(K)$ will denote the set of all points $M(P)$ for all points P of K.

DEFINITIONS. A point P is said to be a limit point of a point-set K if and only if every region that contains P contains at least one point of K distinct from P. The boundary of a point-set K is the set of all points $[X]$ such that every region that contains X contains at least one point of K and at least one point that does not belong to K. If K is a point-set, K' denotes the set of points composed of K plus its boundary. If R is a region the point-set $\overline{S}—R'$ is called the *exterior* of R. A point in the exterior of R is said to be *without* R.

A set of points is said to be *connected* if however it be divided into two mutually exclusive subsets, one of them contains a limit point of the other one. A set of points is said to be *closed* if it contains all its limit points. A set of points is said to be *continuous* if it is both closed and connected.

A *domain* is a set of points K such that if P is a point of K then there exists a region that contains P and is contained in K.

A set of regions G is said to *cover* a point-set K if each point of K belongs to at least one region of the set G. If for every infinite set of regions G that covers the point-set K there exists a finite subset of G that also covers K, then K is said to *possess the Heine-Borel property.*

A set of points K is said to be *bounded* if there exists a region R such that K is a subset of R'.

If A and B are two distinct points, a *simple continuous arc* from A to B is a continuous bounded point-set that contains A and B, but is disconnected† by the omission of any one of its points other than A and B.

A *simple closed curve* is a continuous bounded point-set which is disconnected by the omission of any two of its points.

---

* By a one-to-one transformation of $\overline{S}$ into itself is meant a transformation T such that (1) for each point P of $\overline{S}$ there exists one and only one point $\overline{P}$ of $\overline{S}$ such that T transforms P into $\overline{P}$, (2) for each point $\overline{P}$ of $\overline{S}$ there is one and only one point P of $\overline{S}$ such that T transforms P into $\overline{P}$. *Point* is wholly undefined. *Region* is undefined except in so far as it is understood that every region is *some* sort of collection of points. *Motion* is undefined except in so far as it is understood that every motion is *some* sort of one-to-one transformation of $\overline{S}$ into itself. In addition to this information, no further information (aside from that furnished by the axioms of the system Σ) is presupposed concerning the terms *point, region,* and *motion.*

† A connected point-set K is said to be disconnected by the omission of a proper subset N if $K—N$ is not connected.

§ 3. *The Axioms of* Σ.

Axiom 1. *There exists at least one region.*

Axiom 2. *If* R *and* K *are regions and* R' *is a subset of* K' *then* R *is a subset of* K.

Axiom 3. *If the region* R_1 *contains the point* O *in common with the region* R_2, *there exists a region* R *containing* O *such that* R' *is common to* R_1 *and* R_2.

Axiom 4. *If* R_1 *and* R_2 *are regions and* R_2' *is a subset of* R_1 *then* $R_1 - R_2'$ *is a non-vacuous connected point-set.*

Axiom 5. *If* R_1 *and* R_2 *are regions there exists a region* R *that contains both* R_1' *and* R_2'.

Axiom 6. *Every simple closed curve is the boundary of a region.*

Axiom 7. *If* O *is a point and* L *and* N *are closed bounded point-sets with no point in common, there exists a region* K *containing* O *such that if* P *is a point in* K *then every region that contains both a point of* L *and a point of* N *can be transformed, by a motion that carries some point of* L *into* O, *into a point-set that contains both* O *and* P.

Axiom 8. *If* R *is a region and* M *is a motion then* $M(R)$ *is a region.*

Axiom 9.* *If* A, B, C, A', B', C' *are points, distinct or otherwise, such that every three regions that contain* A, B *and* C, *respectively, can be transformed by some motion into regions containing* A', B', *and* C', *respectively, then there exists a motion that transforms* A *into* A', B *into* B', *and* C *into* C'.

Axiom 10.† *If* M *is a motion there exists a motion* M^{-1} *such that if* $M(A)=B$ *then* $M^{-1}(B)=A$.

Axiom 11.‡ *If* M *and* N *are motions there exists a motion* MN *such that, for every point* P, $M(N(P))=MN(P)$.

Axiom 12.§ *If* R_1 *and* R_2 *are regions bounded respectively by the simple closed curves* J_1 *and* J_2, R_1' *and* R_2' *have no point in common,* A_1, B_1, *and* C_1 *are three distinct points on* J_1, *and* A_2, B_2, *and* C_2 *are three distinct points on* J_2, *and there exist three simple continuous arcs* A_1XA_2, B_1YB_2, *and* C_1ZC_2 *such that no two of these arcs have a point in common and no one of them has any point other than an end-point in common either with* R_1' *or with* R_2' *and* M *is a motion such that* R_1' *and* $M(R_2')$ *have no point in common and*

---

* Cf. Hilbert's Axiom III, *loc. cit.*, p. 169.

† Cf. Hilbert, *loc. cit.*, p. 167.

‡ Cf. Hilbert's Axiom I, *loc. cit.*, p, 167.

§ Cf. J. R. Kline, "A Definition of Sense on Closed Curves in Non-metrical Plane Analysis Situs," *Annals of Mathematics*, Vol. XIX (1918), pp. 185–200. Axiom 12 corresponds to Hilbert's assumption (*loc. cit.*, p. 167) that motion does not change sense on any simple closed curve.

there exist three arcs $A_1\bar{X}M(A_2)$, $B_1\bar{Y}M(B_2)$, and $C_1\bar{Z}M(C_2)$ from A_1 to $M(A_2)$, from B_1 to $M(B_2)$, and from C_1 to $M(C_2)$, respectively, then there exist three such arcs such that no two of them have a point in common and no one of them has any point other other than end-point in common either with R_1' or with $M(R_2')$.

§ 4. *Consequences of Axioms* 1–4 *and* 7–11.

THEOREM 1. *If the point P is a limit point of the point-set K, and M is a motion, then $M(P)$ is a limit point of $M(K)$.*

Proof. If $M(P)$ were not a limit point of $M(K)$ there would exist a region R containing $M(P)$, but no point of $M(K)$ other than $M(P)$. But in this case $M^{-1}(R)$ would be a region containing P but no point of K other than P, contrary to the hypothesis that P is a limit point of K.

THEOREM 2. *No point of a region is a boundary point of that region.*

THEOREM 3. *Every region contains infinitely many points.*

Theorem 3 can be easily proved with the use of Axioms 3 and 4.

THEOREM 4. *If A and B are distinct points, and C is any point whatever, there exists a region containing C which can not be transformed by a motion into a point-set K such that K' contains both A and B.*

Proof. If no region contains the point C then it is vacuously true that if R_1, R_2, and R_3 are three regions containing C there exists a motion $\bar{M}$ such that $\bar{M}(R_1)$ contains A, and $\bar{M}(R_2)$ and $\bar{M}(R_3)$ contain B. It follows by Axiom 9 that there exists a motion that carries C into both A and B. Thus the supposition that there is no region containing C leads to a contradiction. Suppose that every region containing C can be transformed by a motion into a point-set K such that K' contains both A and B. By Axiom 3, if R is a region containing C, there exists a region $\bar{R}$ containing C such that $\bar{R}'$ is a subset of R. By hypothesis there exists a motion M such that $[M(\bar{R})]'$ contains both A and B. By Theorem 1 $[M(\bar{R})]'=M(\bar{R}')$. Hence $M(R)$ contains both A and B. It follows by Axiom 9 that there exists a motion that transforms C into both A and B. Thus the supposition that Theorem 4 is false leads to a contradiction.

THEOREM 5. *If L and N are two closed, bounded point-sets with no point in common, and O is any point whatever, there exists a region R containing O such that R' can not be transformed by a motion into a point-set that contains both a point of L and a point of N.*

Proof. By Axiom 7 and Theorem 3 there exists a point P distinct from O such that every region that contains both a point of L and a point of N can be transformed by a motion into a point-set that contains both O and P. By Theorem 4 there exists about* O a region $\bar{R}$ which can not be moved into a point-set containing both O and P. If there should exist a motion M such that $M(\bar{R})$ contains both a point of L and a point of N, then there would exist a motion $\bar{M}$ such that $\bar{M}M(\bar{R})$ contains both O and P. But this would involve a contradiction. By Axiom 3 there exists about O a region R such that R' is a subset of $\bar{R}$. The region R clearly satisfies the condition of Theorem 5.

THEOREM 6. *If P is a limit point of $M+N$ it is a limit point either of M or of N.*

Theorem 6 can be proved with the use of Axiom 3.

THEOREM 7. *If the point P is a limit point of the point-set M, then every region that contains P contains infinitely many points of M.*

Proof. Suppose the region R contains the point P and has in common with M only a finite set of points $P_1, P_2, P_3, \ldots, P_n$ distinct from P. By Theorem 4 for each $i (1 \leqq i \leqq n)$ there exists about P a region R_i that can not be transformed by a motion into a point-set containing P and P_i. But there exists a motion that leaves all points fixed. This motion carries R_i into R_i. But R_i contains P. It follows that R_i does not contain P_i. The regions R_1 and R_2 contain in common a region K_2 containing P. The region K_2 contains neither P_1 nor P_2. Similarly there exists in K_2 a region K_3 that contains P, but no one of the points P_1, P_2, P_3. This process may be continued. It follows that there exists about P a region K_n that lies in R but contains no point of the set $P_1, P_2, P_3, \ldots, P_n$. Hence P is not a limit point of M. Thus the supposition that Theorem 7 is false leads to a contradiction.

THEOREM 8. *If the region R contains a point O in the region K and a point P without K, then it contains a point on the boundary of K.*

Proof. By Axiom 3 there exists about O a region R_1 which is a subset both of R and of K. By Theorem 3 there exists in R_1 a point $\bar{P}$ distinct from O. By Theorem 7 and Axiom 3 there exists about O a region R_2 such that R_2' is a subset of $R_1 - \bar{P}$. By Axiom 4 $R - R_2'$ is connected. But it contains the point $\bar{P}$ in K and the point P without K. Hence it contains a point of the boundary of K.

---

* In this connection "about" is synonymous with "containing."

THEOREM 9. *If O is a point there exists a countably infinite sequence of regions $R_1, R_2, R_3, \ldots,$ such that* (1) *P is the only point they have in common,* (2) *if n is a positive integer and M is a motion such that $M(R_{n+1})$ contains O, then $M(R'_{n+1})$ is a subset of R_n,* (3) *if R is a region containing O there exists an n such that R_n is a subset of R.*

Proof. By Axiom 1 and Theorems 3 and 4 there exists a region K_1 containing O. By Axiom 3 there exists a region K_2 containing O such that K'_2 is a subset of K_1 and a region K_3 containing O such that K'_3 is a subset of K_2. It follows by two applications of Axiom 4 that $K_1 - K'_3$ is a connected point-set containing at least two distinct points. Let $\overline{P}$ denote a definite point of $K_1 - K'_3$. Then $\overline{P}$ is a limit point of $K_1 - K'_3 - \overline{P}$. It follows that there exists a countable sequence of points $\overline{P}_1, \overline{P}_2, \overline{P}_3, \ldots$ all distinct from $\overline{P}$ such that $\overline{P}$ is a limit point of the point-set $\overline{P}_1 + \overline{P}_2 + \overline{P}_3 + \ldots$. By Theorem 7 and Axiom 7 there exists a motion $\overline{M}$ such that $\overline{M}(\overline{P}) = O$. For every n let P_n denote $\overline{M}(\overline{P}_n)$. Then O is a limit point of the point-set $P_1 + P_2 + P_3 + \ldots$. By Theorem 4 there exists about O a region R_1 such that if M is a motion that transforms R_1 into a point-set containing O then $M(R'_1)$ does not contain P_1. By Theorems 6, 7 and 5 there exists a region R_2 containing O such that if M is a motion that transforms R_2 into a point-set containing O, then $M(R'_2)$ contains no point of the closed, bounded point-set $P_2 + L_1$ where L_1 is the boundary of R_1. It follows by Theorem 8 that, for every such motion M, $M(R'_2)$ is a subset of R_1. This process may be continued. It follows that there exists a sequence of regions $R_1, R_2, R_3, \ldots$ containing O such that if n is a positive integer and M is a motion that transforms R_{n+1} into a point-set containing O, then $M(R'_{n+1})$ contains no point of the point-set $P_1 + P_2 + P_3 + \ldots + P_n + S - R_n$. The sequence $R_1, R_2, R_3, \ldots$ satisfies the requirements of Theorem 9. That it satisfies requirement (2) is obvious. Suppose it does not satisfy both (1) and (3). Then there exists a point P distinct from O and a region R containing O such that for every n R_n contains a point of the closed and bounded point-set $P + L$ where L is the boundary of R. It follows by Axiom 7 that there exists a positive integer m such that, for every n, R_n can be transformed by some motion into a point-set containing P_m and O. But R_{m+1} can not be moved into such a point-set. Thus the supposition that $R_1, R_2, R_3, \ldots$ does not satisfy requirements (1) and (3) has led to a contradiction.

THEOREM 10. *Every region is a connected set of points.*

Proof. Suppose R_1 is a region. There exists in R_1 a point P. By Theorem 9 there exists a sequence of regions $R_2, R_3, R_4, \ldots,$ all lying in R_1

and having in common only the point P and such that (1) for each n, R'_{n+1} is a subset of R_n, (2) if R is a region containing P there exists an n such that R contains R_n. By Axiom 4, $R_1-R'_n$ is connected. But

$$R_1=(R_1-R'_2)+(R_1-R'_3)+(R_1-R'_4)+\ldots .+P,$$

$R_1-R'_n$ is a subset of $(R_1-R'_{n+1})$ and P is a limit point of

$$(R_1-R'_2)+(R_1-R'_3)+(R_1-R'_4)+\ldots .$$

It easily follows that R_1 is connected.

THEOREM 11. *Every boundary point of a region is a limit point of the exterior of that region.*

Proof. Suppose the boundary of the region R contains a point X which is not a limit point of $\bar{S}-R'$. Then there exists a region $\bar{R}$ that contains X and lies wholly in R'. It follows that $\bar{R}'$ is a subset of R'. Therefore, by Axiom 2, $\bar{R}$ is a subset of R. Thus X belongs to R and is therefore not a boundary point of R.

THEOREM 12. *If $R_1, R_2, R_3, \ldots$ is a sequence of regions closing down* on the point O, $M_1, M_2, M_3, \ldots$ are motions and L and N are two closed and bounded point-sets with no point in common, then there do not exist infinitely many positive integers n such that $M_n(R'_n)$ contains both a point of L and a point of N.*

Theorem 12 is a consequence of Theorem 5.

THEOREM 13. *If $R_1, R_2, R_3, \ldots$ is a sequence of regions closing down on the point O, $M_1, M_2, M_3, \ldots$ are motions, and A_1, A_2, A_3, ... and $B_1, B_2, B_3, \ldots$ are two infinite sequences of points such that, for every n, A_n and B_n are both in M_n (R_n), then if $A_1+A_2+A_3+\ldots$ has a limit point every such point is also a limit point of $B_1+B_2+B_3+\ldots$.*

Proof. Suppose X is a limit point of $A_1+A_2+A_3+\ldots$, and R is a region containing X. By Axiom 3 there exists a region $\bar{R}$ containing X such that $\bar{R}'$ is a subset of R. The region $\bar{R}$ contains infinitely many distinct points $A_{n_1}, A_{n_2}, A_{n_3}, \ldots$ of the sequence $A_1, A_2, A_3, \ldots$. It follows with the help of Theorems 8 and 12 that, for infinitely many positive integers i, R_{n_i}, and therefore B_{n_i}, is a subset of R. It follows that X is a limit point of $B_1+B_2+B_3+\ldots$.

THEOREM 14. *Every bounded infinite set of points has at least one limit point.*

*A sequence of regions $R_1, R_2, R_3, \ldots$ is said to *close down on* the point O if it satisfies with respect to O all the requirements of Theorem 9.

Proof. Suppose that R is a region and that $X_1, X_2, X_3, \ldots$ is an infinite set of distinct points lying in R' and having no limit point. Let $R_1, R_2, R_3, \ldots$ denote a sequence of regions closing down on a point O. For each n there exists a motion M_n such that $M_n(O)=X_n$. For each n there exists in the region $M_n(R_n)$ a point $\bar{X}_n$ distinct from every point of the set $X_1, X_2, X_3, \ldots$ and lying in R'. By Theorem 13 the point-set $\bar{X}_1+\bar{X}_2+\bar{X}_3+\ldots$ has no limit point. Thus $X_1+X_2+X_3+\ldots$ and $\bar{X}_1+\bar{X}_2+\bar{X}_3+\ldots$ are closed, bounded point-sets with no point in common. But, for every n, $M_n(R_n)$ contains a point of each of these sets. This is contrary to Theorem 12.

THEOREM 15. *There does not exist a compact* point-set K and an uncountably infinite set G of mutually exclusive regions such that every region of the set G contains a point of K.*

Proof. Suppose there does exist a compact point-set K and an uncountable set of regions G satisfying such conditions. Let $R_1, R_2, R_3, \ldots$ denote a sequence of regions closing down on some point O. If X is a point of K lying in a region g of the set G then for each n there exists a motion M_n such that $M_n(O)=X$. By Theorems 8 and 12 there exists a positive integer m such that $M_m(R_m)$ is a subset of g. It follows by Zermelo's Postulate that there exists a set $\bar{G}$ of regions such that (1) each region of the set G contains one and only one region of the set $\bar{G}$, (2) for each region $\bar{g}$ of the set $\bar{G}$ there exists a motion that transforms some region of the set $R_1, R_2, R_3, \ldots$ into $\bar{g}$ and transforms the point O into a point of K that lies in $\bar{g}$. In view of the fact that G is an uncountable set it follows that there exists a positive integer n and a set of motions $M_1, M_2, M_3, \ldots$ such that, for every j, $M_j(O)$ is a point of K and such that no two of the regions $M_1(R_n), M_2(R_n), M_3(R_n), \ldots$ have a point in common. By hypothesis the set of points $M_1(O)+M_2(O)+M_3(O)\ldots$ has at least one limit point Z. There exists a motion $\bar{M}$ such that $\bar{M}(R_{n+1})$ contains Z. There exists k such that $M_k(O)$ is in $\bar{M}(R_{n+1})$. Hence O is in $M_k^{-1}\bar{M}(R_{n+1})$. It follows that $M_k^{-1}\bar{M}(R'_{n+1})$ is a subset of R_n. Hence $\bar{M}(R'_{n+1})$ is a subset of $M_k(R_n)$. Therefore $M_k(R_n)$ contains Z. Hence there exists an index i distinct from k such that $M_k(R_n)$ contains $M_i(O)$. It follows that $M_k(R_n)$ and $M_i(R_n)$ have a point in common. Thus the supposition that Theorem 15 is false leads to a contradiction.

THEOREM 16. *Every compact set of points is a subset of a compact domain.*

* A set of points K is said to be *compact* if every infinite subset of K has at least one limit point. Cf. M. Fréchet, "Sur quelques points du calcul fonctionnel," *Rendiconti del circolo matematico di Palermo*, Vol. XXII (1906), p. 6.

Proof. Let $R_1, R_2, R_3, \ldots$ be a sequence of regions closing down on a point O. There exists an index m greater than 1 such that if M is a motion such that $M(R_m)$ contains a point of R_m then $M(R_m)$ is a subset of R_1. Suppose there exists a compact set of points K which is not a subset of a compact domain. Then there must exist an infinite set of distinct points $P_1, P_2, P_3, \ldots$ lying in K, an infinite set of distinct points $\overline{P}_1, \overline{P}_2, \overline{P}_3, \ldots$ such that $\overline{P}_1+\overline{P}_2+\overline{P}_3+\ldots$ has no limit point and a set of motions $M_1, M_2, M_3, \ldots$ such that for every n the region $M_n(R_m)$ contains both P_n and $\overline{P}_n$. The point-set $P_1+P_2+P_3$ has at least one limit point $\overline{O}$. There exists a motion $\overline{M}$ such that $\overline{M}(O)=\overline{O}$. There exists an infinite set of distinct positive integers $n_1, n_2, n_3, \ldots$ such that the points $P_{n_1}, P_{n_2}, P_{n_3}, \ldots$ are all in $\overline{M}(R_m)$. Each of the regions $M_{n_1}(R_m), M_{n_2}(R_m), M_{n_3}(R_m), \ldots$ is a subset of $\overline{M}(R_1)$. Hence the infinite point-set $\overline{P}_{n_1}+\overline{P}_{n_2}+\overline{P}_{n_3}+\ldots$ is a subset of $\overline{M}(R_1)$. It follows by Theorem 14 that $\overline{P}_1+\overline{P}_2+\overline{P}_3+\ldots$ has a limit point. Thus the supposition that Theorem 16 is false has led to a contradiction.

THEOREM 17. *If the compact point-set T is covered by an uncountably infinite set of regions G, then T is covered by some countable subset of G.*

Proof. By Theorem 16 the compact point-set T is a subset of some compact domain K. Let $R_1, R_2, R_3, \ldots$ denote a sequence of regions closing down on a point O. For each point P of T there exists an integer n_P greater than 1, and a motion M_P such that the region $M_P(R_{n_P-1})$ contains P and lies both in K and in some region of the set G. Let $\overline{G}$ be the set of all regions $M_P(R_{n_P})$ for all points P of T. There exists a finite or countably infinite set of distinct positive integers $m_1, m_2, m_3, \ldots$ and a sequence $\overline{G}_{m_1}, \overline{G}_{m_2}, \overline{G}_{m_3}, \ldots$ of subsets of $\overline{G}$ such that (1) $\overline{G}=\overline{G}_{m_1}+\overline{G}_{m_2}+\overline{G}_{m_3}+\ldots$ (2) for every i, $\overline{G}_{m_i}$ is the set of all regions $M_P(R_{n_P})$ for which $n_P=m_i$. For every i let T_i denote the set of all those points of T which are covered by the set of regions $\overline{G}_{m_i}$. The regions of the set $\overline{G}_{m_i}$ can be arranged in a well-ordered sequence $g_{i1}, \bar{g}_{i2}, \bar{g}_{i3}, \ldots \bar{g}_{i\omega}, \ldots$. Call this well-ordered sequence β. Let g_{i2} be the first region in the sequence β that contains a region that has no point in common with g_{i1}. Let g_{i3} be the first one following g_{i1} that contains a region that has no point in common with g_{i1} or with g_{i2}. Continue this process thus obtaining a well-ordered sequence γ such that (1) every element of γ is an element of β, (2) if γ_1 is any subsequence of γ such that there exists an element of γ that follows all the elements of γ_1, then the first element of γ that follows all the elements of γ_1 is the first element of β that contains a region that has no point in common with any of the elements of γ_1. There is not an uncountable

infinity of elements in the sequence γ. For if there were, there would exist in K an uncountable infinity of distinct regions, no two of which have a point in common, which is contrary to Theorem 15. It follows that there exists a countable subset G_i of the set of regions $\overline{G}_{m_i}$ such that every point of T_i either is in a region of the set G_i or is a limit point of the point-set L_i obtained by adding together all the regions $M_{P_1}(R_{m_i})$, $M_{P_2}(R_{m_i})$, $M_{P_3}(R_{m_i})$, ... of the set G_i. Let $\overline{\overline{G}}_i$ denote the set of regions $M_{P_1}(R_{m_i-1})$, $M_{P_2}(R_{m_i-1})$, $M_{P_3}(R_{m_i-1})$, ... The set $\overline{\overline{G}}_i$ covers T_i. For suppose there exists a point X of T_i which lies in no region of the set $\overline{\overline{G}}_i$. Then X is a limit point of L_i. There exists a positive integer k_i such that the region R_{k_i} can not be transformed by a motion into a point-set containing both a point in R_{m_i} and a point in $\overline{S}-R_{m_i-1}$. But there exists a motion $\overline{M}$ such that $\overline{M}(R_{k_i})$ contains X. The region $\overline{M}(R_{k_i})$ contains a point of L_i and therefore, for some j, a point in common with $M_{P_j}(R_{m_i})$. If it also contained a point in common with $\overline{S}-M_{P_j}(R_{m_i-1})$ then $M_{P_j}^{-1}\overline{M}(R_{k_i})$ would contain a point of R_{m_i} and a point of $\overline{S}-R_{m_i-1}$. But this is impossible. It follows that, for every i, T_i is covered by $\overline{\overline{G}}_i$. But every region of $\overline{\overline{G}}_i$ is a subset of some region of G, and T is the sum of the finite or countably infinite set of point-sets T_1, T_2, T_3, ... It follows that T is covered by a countable subset of G.

THEOREM 18. *If a closed and compact point-set is covered by an infinite set of regions then it is also covered by some finite subset of that set of regions.*

Proof. By Theorem 17 if a compact point-set is covered by an infinite set of regions G it is covered by a countable subset of G. But if a closed and compact point-set is covered by a *countable* set of regions then* it is covered by a finite subset of that countable set.

THEOREM 19. *If H' is a closed and bounded set of points there exists an infinite sequence of regions $K_{H'1}$, $K_{H'2}$, $K_{H'3}$, ... such that* (1) *if m is a positive integer and P is a point of H', there exists an integer n, greater than m, such that $K_{H'n}$ contains P,* (2) *if P and $\overline{P}$ are distinct points of H' lying in a region R, there exists an integer δ such that if $n > \delta$ and $K_{H'n}$ contains P then $K'_{H'n}$ is a subset of $R-\overline{P}$.*

Proof. Let R_1, R_2, R_3, ... be a set of regions closing down on a point O. If X is a point of H' and n is a positive integer, the region R_n can be transformed by a motion into a region K_{Xn} containing X. For any fixed n consider

---

* Cf. F. Hausdorff, "Grundzüge der Mengenlehre," Veit & Co., Leipzig, 1914, p. 231.

the set of all such K_{Xn}'s for all points X of H'. By Theorems 14 and 18 there exists a finite number of these K_{Xn}'s, say $K_n^{(1)}, K_n^{(2)}, K_n^{(3)}, \ldots, K_n^{(m_n)}$ covering H'. Let n take on successively all positive integral values. Let $K_{H'1}$, $K_{H'2}$, $K_{H'3}, \ldots, K_{H'm_1}$ denote $K_1^{(1)}, K_1^{(2)}, K_1^{(3)}, \ldots, K_1^{(m_1)}$, respectively. For $n \geqq 1$ let $K_{H', m_n+1}, K_{H', m_n+2}, K_{H', m_n+3}, \ldots, K_{H'm_{n+1}}$ denote $K_{n+1}^{(1)}, K_{n+1}^{(2)}, K_{n+1}^{(3)}, \ldots, K_{n+1}^{m_{n+1}}$, respectively. The infinite sequence of regions $K_{H'1}, K_{H'2}, K_{H'3}, \ldots$ so obtained satisfies all the requirements of Theorem 19.

Theorem 20. *If H is a region there exists an infinite sequence of regions $K_{H_1}, K_{H_2}, K_{H_3}, \ldots$ such that each K_{Hn} is a subset of H, and such that if, in the statement of Conditions* (1) *and* (2) *of Theorem* 19, *H' is replaced by H, the resulting conditions are fulfilled.*

Proof. Let $K_{H'1}, K_{H'2}, K_{H'3}, \ldots$ be a sequence of regions satisfying all the requirements of Theorem 19. Let K_{H1} denote the first region of this sequence that lies in H. Let K_{H2} denote the first region that follows K_{H1} in the same sequence and lies in H. If this process is continued there will be obtained a sequence of regions satisfying the requirements of Theorem 20.

§ 5. *Consequences of Axioms* 1–12.

Theorem 21. *If R is a region $\overline{S}$—R' is a connected set of points.*

Proof. That $\overline{S}$—R' exists and contains infinitely many points may be easily proved with the use of Axioms 5 and 4. Suppose A and B are two points in $\overline{S}$—R'. There exist regions R_A and R_B containing A and B, respectively. By two applications of Axiom 5 there exists a region K that contains R', R'_A and R'_B, and therefore contains A and B. By Axiom 4, K—R' is connected. Thus every two points of S—R' lie together in a connected subset of S—R'. It easily follows that S—R' is connected.

Theorem 22. *There exists an infinite set of points that has no limit point.*

Proof. Suppose on the contrary that every infinite set of points has at least one limit point. Then $\overline{S}$ is compact and closed. But every point of $\overline{S}$ is in some region. Hence by Theorem 18 there exists a finite set of regions R_1, $R_2, R_3, \ldots, R_n$, such that every point is in some region of this set. But by n applications of Axiom 5 there exists a region R that contains all the regions $R_1, R_2, R_3, \ldots, R_n$. Hence R contains all points. But, by Axiom 5, there exists a region K that contains R' and, by Axiom 4, K contains at least one point that is not in R. Thus the supposition that Theorem 22 is false has led to a contradiction.

With the use in particular of Axioms 3 and 5, and Theorems 14, 10 and 20, the truth of the following theorem may be established by methods in large part similar to or identical with those employed in the proof of Theorem 15 on pages 136–139 of my paper "On the Foundations of Plane Analysis Situs." *

THEOREM 23. *Every two points of a connected domain are the extremities of a simple continuous arc that lies wholly in that domain.*

THEOREM 24. *If O is a point in a region R there exists a simple closed curve that lies in R and encloses O.*

Proof. There exists a region $\bar{\bar{R}}$ which can not be transformed by a motion into a region L such that L' contains both O and a point of $S—R$. By Theorems 23 and 10, if A and B are two points in the region $\bar{\bar{R}}$, there exists an arc AXB lying wholly in $\bar{\bar{R}}$. There exists a region K containing X such that K' is a subset of $\bar{\bar{R}}—A—B$. There exists an arc AYB that lies wholly in the domain $\bar{\bar{R}}—K'$. It is clear † that the point-set composed of the arcs AXB and AYB together contains at least one simple closed curve J. By Axioms 6 and 2, and Theorem 21, I, the interior of J, is a subset of $\bar{\bar{R}}$. There exists a motion M such that $M(I)$ contains O. The closed curve $M(J)$ lies in R and encloses O.

THEOREM 25. *If J and C are simple closed curves, O is a point on J but not on C, A_1 and A_2 are distinct points common to C and J, and A_1XA_2 is an arc on C such that $\underset{\smile}{A_1XA_2}$ ‡ lies within J, then there exist two points O_1 and O_2 distinct from O such that*

(1) *O_1 and O_2 lie on the intervals A_1O and A_2O of the arc A_1OA_2 of the closed curve J.*

(2) *There is on the curve C an arc O_1YO_2 such that $\underset{\smile}{O_1YO_2}$ is within J.*

(3) *If B_1 and B_2 are points on the intervals O_1O and O_2O, respectively, of the arc A_1OA_2 of J, such that there exists on C from B_1 to B_2 an arc which, except for its end-points, lies entirely within J, then $B_1=O_1$ and $B_2=O_2$.*

For a proof of Theorem 25 see pages 152 and 153 of F. A.

---

* *Transactions of the American Mathematical Society*, Vol. XVII (1916), pp. 131–164. Hereafter this paper will be referred to as F. A.

† For a proof that simple continuous arcs and simple closed curves as defined in the present paper have certain fundamental properties including properties of linear and cyclical order respectively, see my paper, "Concerning Simple Continuous Curves." See also F. A., pp. 139 and 140.

‡ If ABC is an arc, $\underset{\smile}{ABC}$ denotes the point-set $ABC—A—B$. Likewise if AB is an arc, $\underset{\smile}{AB}$ denotes the point-set $AB—A—B$.

THEOREM 26. *If P is a point on a closed curve J, and $\bar{J}$ is a closed curve enclosing P and containing at least one point within J, then there exist two closed curves Q and $\bar{Q}$ such that* (1) *every point of Q belongs either to J or to $\bar{J}$ and so does every point of $\bar{Q}$,* (2) *the curves Q and $\bar{Q}$ contain in common a segment of J that contains P,* (3) *the interiors of Q and $\bar{Q}$ are both subsets of the interior of $\bar{J}$,* (4) *every point within Q is within J and every point within $\bar{Q}$ is without J.*

Proof. If the curve J had no point without $\bar{J}$, the interior of J would be a subset of the interior of $\bar{J}$ and therefore could not contain a point of $\bar{J}$. It follows that J contains a point C without $\bar{J}$. The curve J is the sum of two simple continuous arcs PEC and PFC. Let A and B denote the first points that the arcs PEC and PFC, respectively, have in common with $\bar{J}$. The curve $\bar{J}$ is the sum of two arcs $A\bar{E}B$ and $A\bar{F}B$ (Fig. 1). Let H denote the interior of

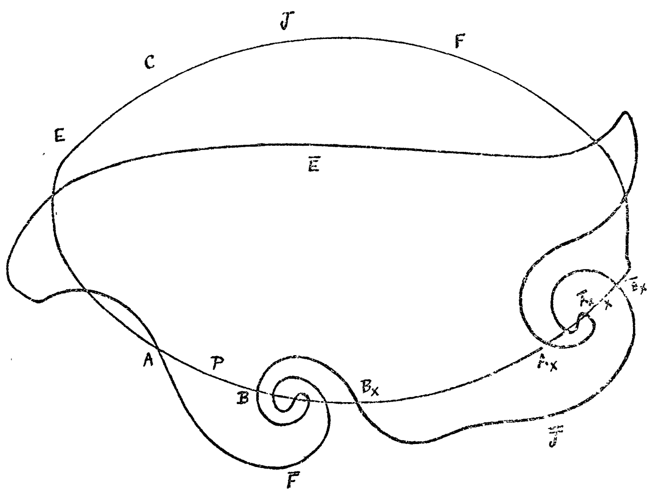

FIG. 1.

the closed curve $A\bar{E}BPA$ formed by the arc $A\bar{E}B$ and the interval APB of the curve J, and let K denote the interior of the closed curve $A\bar{F}BPA$ formed by the arc $A\bar{F}B$ and the same interval APB. By Theorems 24 and 25 of F. A. the point-sets H, K and APB are mutually exclusive and $H+K+\underset{\smile}{APB}=\bar{I}$, the interior of $\bar{J}$. If the curve J contains a point X in H, then J has in common with the curve $A\bar{E}BPA$ two distinct points $\bar{A}_x$ and $\bar{B}_x$ such that the segment $\bar{A}_xX\bar{B}_x$ of J is a subset of H. By Theorem 25 there exists on J an arc A_xB_x such that (1) A_x is on the subinterval $\bar{A}_xP$ of the interval $\bar{B}_x\bar{A}_xP$ of the curve $A\bar{E}BPA$, (2) B_x is on the subinterval $\bar{B}_xP$ of the interval $\bar{A}_x\bar{B}_xP$ of $A\bar{E}BPA$, (3) $\underset{\smile}{A_xB_x}$ is in H, (4), if Y is a point on the subinterval A_xP of the interval

$\overline{B}_x\overline{A}_xP$ of the curve $A\overline{E}BPA$, and Z is a point on the subinterval B_xP of the interval $\overline{A}_x\overline{B}_xP$ of $A\overline{E}BPA$ such that the interval ACB of J contains a segment that has Y and Z as end-points and lies entirely in H, then $Y=A_x$ and $Z=B_x$. It is clear that the arc A_xB_x is completely determined by the point X. For every X on ACB that lies in H construct the corresponding A_xB_x. Let $\overline{h}$ denote the point-set which is composed of all the A_xB_x's so constructed together with every point F on $A\overline{E}BPA$ which has the property that for no X is F separated from P by A_x and B_x. It may be easily proved that $\overline{h}$ is a simple closed curve. That $\overline{H}$, the interior of $\overline{h}$, is a subset of H, and therefore of the interior of $\overline{J}$ is a consequence of Axioms 6 and 2 and Theorem 21. It is clear that $\overline{h}$ contains APB and that $\overline{H}$ contains no point of J. Similarly there exists a closed curve $\overline{k}$ containing APB such that every point of $\overline{k}$ belongs either to J or to $A\overline{F}BPA$, and such that (1) $\overline{K}$, the interior of $\overline{k}$, is a subset of K, (2) $\overline{K}$ contains no point of J. The point-set $\overline{h}+\overline{k}-\overset{\frown}{ABP}$ is evidently a simple closed curve α. If $\overset{\frown}{APB}$ were in the exterior of α, then by Theorem 27 of F. A., either $\overline{H}$ would be a subset of $\overline{K}$, or $\overline{K}$ would be a subset of $\overline{H}$, neither of which is possible in view of the facts that $\overline{H}$ and $\overline{K}$ are subsets of H and K respectively, and H and K have no point in common. It follows that APB is within α. Hence, by Theorem 25 of F. A., R, the interior of α, $=APB+\overline{H}+\overline{K}$. Since R contains P, a boundary point of J, it must contain a point P_1 within J and a point P_2 without J. Let Q denote that one of the curves $\overline{h}$ and $\overline{k}$ that encloses P_1 and let $\overline{Q}$ denote the other one. Since the interior of Q contains a point within J and no point on J it must lie entirely within J. Hence P_2 is within $\overline{Q}$, and $\overline{Q}$ is entirely without J. The curves Q and $\overline{Q}$ fulfill all the requirements of Theorem 26.

THEOREM 27. *If R and $\overline{R}$ are Jordan regions,* and P is a point in $\overline{R}$ and on the boundary of R, there exist in $\overline{R}$ two Jordan regions K and $\overline{K}$ such that $\overline{K}$ contains P, K lies in R, and all those points of the boundary of R that lie in $\overline{K}$ are points also of the boundary of K.*

THEOREM 28. *If R and $\overline{R}$ are Jordan regions and P is a point in $\overline{R}$ and on the boundary of R, there exist in $\overline{R}$ two regions L and $\overline{L}$ such that $\overline{L}$ contains P, L lies in $S-R'$ and all those points of the boundary of R that lie in $\overline{L}$ are points also of the boundary of L.*

Proofs of Theorems 27 and 28. Let J and $\overline{J}$ denote the boundaries of R and $\overline{R}$ respectively, and let K and L respectively denote the interiors of the

* A *Jordan* region is the interior of a simple closed curve.

curves Q and $\bar{Q}$ that satisfy with respect to J, $\bar{J}$ and P the requirements of Theorem 26. Let $\bar{K}$ and $\bar{L}$ denote two Jordan regions both of which contain P and lie in $\bar{R}$, but neither of which contains any point of J that is not on the interval APB described in the above proof of Theorem 26. It is clear that the regions K and $\bar{K}$ satisfy the requirements of Theorem 27 and that L and $\bar{L}$ satisfy those of Theorem 28.

If Theorems 20, 10, 21, 18 and 14, 22, 26, 27 and Axiom 6 of the present paper are compared with Axioms 1–8 of F. A., it will be seen that if the term *region* employed in the latter axioms is restricted to mean Jordan region, the so modified axioms hold true* for the set of all points S lying in a given region R of our space $\bar{S}$. Furthermore, in view of Theorem 24 it is clear that if a point is a limit point of a point-set in accordance with the definition given in the present paper it is also a limit point of that point-set in accordance with the definition obtained by substituting "Jordan region" for "region" in that definition and conversely. In view of a theorem established in my paper "Concerning a Set of Postulates for Plane Analysis Situs,"† it follows that the following theorem holds true.

THEOREM 29. *If R is a region, then between the points of R and the set of all sensed pairs of real numbers there is a one-to-one correspondence which is continuous in the sense that in R the point P is a sequential limit point‡ of the sequence of points $P_1, P_2, P_3, \ldots.$ if and only if $\lim_{n=\infty} x_n = x$ and $\lim_{n=\infty} y_n = y$ where (x_n, y_n) and (x, y) are the number pairs that correspond to P_n and to P respectively.*

Definitions. If A is a point, a rotation about A is a motion M such that $M(A)=A$. If A and B are two distinct points the set of all points $[X]$ such that B can be transformed into X by a rotation about A is a *circle.* The point A is the *center* of this circle. The notation k_{AB} will be used to denote the circle which contains B and has its center at A. A circle will be called *proper* if it is a simple closed curve and its center lies within it. A proper circle k_{OA} is said to have a *proper interior* if for every point P, within k_{OA} and distinct from O, the circle k_{OP} is proper.

---

* Of course here Axiom 5 would be interpreted to mean that there exists in the region R an infinite set of points that has no limit point in R, and Axiom 2 would be interpreted as meaning that if J is a simple closed curve lying in R the interior of J is connected.

† *Transactions of the American Mathematical Society,* Vol. XX (1919), pp. 169–178.

‡ The point P is said to be a *sequential limit point* of the infinite sequence of points $P_1, P_2, P_3, \ldots.$ if and only if for every region K containing P there exists an integer δ such that if $n > \delta$ then P_n lies in K.

THEOREM 30.* *If O is a point there exists a simple closed curve J enclosing O such that if P is a point on or within J the circle k_{OP} contains infinitely many points.*

Proof. There exists a point A distinct from O and a region R_A containing A such that R'_A does not contain O. By Axioms 7 and 3, and Theorems 5 and 7 there exists a region R containing O such that if P is a point in R then every region that contains both A and O can be transformed by some motion into a region containing P, but such that R can not be transformed by a rotation about O into a point-set containing a point of R'_A. There exists a simple closed curve J that encloses O and lies wholly in R. Let P denote a definite point within or on J but distinct from O. There exists about O, and within J, a region K such that K' can not be transformed by a rotation about O into a point-set that contains P. It may bè easily established with the aid of Theorem 29 that there exists an infinite set $g_1, g_2, g_3, \ldots$ of distinct regions each containing O and A such that if n_1 and n_2 are two distinct positive integers, every point that is common to g_{n_1} and g_{n_2} is either in R_A or in K. For each n there exists a motion M_n and a point P_n in g_n such that $M_n(P_n)=P$. For each n, P_n is in the exterior both of R_A and of K. Hence the points P_1, $P_2, P_3, \ldots$ are all distinct. But by Axiom 10 these points are all on the circle k_{OP}.

With the help of Theorems 29 and 30 the next two theorems can be proved by methods wholly or in large part identical with those employed by Hilbert † in his proofs of more or less closely related theorems.

THEOREM 31. *If O is a point in a region R and there exists a region $\bar{R}$ such that every rotation about O throws R into a subset of $\bar{R}$, and every circle with center at O that contains a point of R' contains infinitely many points, then if P is a point of R' distinct from O the circle k_{OP} is a proper circle with proper interior.*

THEOREM 32. *If k is a proper circle with center at O and with a proper interior then* (1) *every rotation about O that leaves fixed one point within k leaves fixed every point within k,* (2) *if X is a point within k, and M_1 and M_2 are rotations about O such that $M_1(X)=M_2(X)$ then, for every point P within or on k, $M_1(P)=M_2(P)$.*

---

* Cf. Hilbert's Axiom II, *loc. cit.*, p. 168.

† Cf. in particular pp. 173–191, *loc. cit.* On page 179, in lines 5 and 6, Hilbert says "so ist W* offenbar ein Weg, welche K_3 und K_4 innerhalb dieser neuen Kurve verbindet." It is possible that W* should lie *without* this "neuen Kurve." This possibility does not, however, in any way invalidate what follows.

THEOREM 33. *If $\bar{k}$ is a proper circle with a proper interior there exists a proper circle k^* with a proper interior such that k^* and $\bar{k}$ are concentric and $\bar{k}$ is within k^*.*

Proof. Let O denote the center of $\bar{k}$. There exists a region R such that $\bar{k}$ and its interior are subsets of R. If P is a point on $\bar{k}$ there exists about P a region $\bar{R}$ which does not contain O and which can not be transformed by any motion into a region containing both a point of $\bar{k}$ and a point on the boundary of R. There exists about P a region $\bar{K}$ such that if X and Z are two distinct points in $\bar{K}$, the circle k_{XZ} is a simple closed curve enclosing X. The regions $\bar{R}$ and $\bar{K}$ contain in common a region K that contains P. There exist two points A and B on $\bar{k}$ and an arc AXB, lying, except for its end-points, entirely without $\bar{k}$, such that the closed curve α formed by the arc AXB and the arc APB of the circle $\bar{k}$ lies entirely within K.

Suppose that, for some point W within the closed curve α, the circle k_{OW} contains only a finite number of points. Then there must exist a motion $\bar{M}$ such that $\bar{M}(O)=O$, and such that the circles $\bar{M}(k_{WP})$ and $\bar{k}$ have in common an uncountable set of points N such that if X and Y are any two points of N, there exists a motion M such that $M(O)=O$, $M(X)=Y$ and $M(\bar{W})=\bar{W}$ where $\bar{W}$ is the center of $\bar{M}(k_{WP})$. It follows that there exists in the set N two points C and D such that if M_0 is a motion, such that $M_0(C)=D$, $M_0(O)=O$, and $M_0(\bar{W})=\bar{W}$, then there exists no positive integer n such that $M_0^n(C)=C$. It follows that if L denotes the set of all points $[\bar{P}]$ such that, for some positive integer m, $M_0^m(C)=\bar{P}$ then L is everywhere dense both on $\bar{k}$ and on $\bar{M}(k_{WP})$. It follows that $\bar{M}(k_{WP})$ is identical with $\bar{k}$. But this is impossible in view of the fact that $\bar{M}(k_{WP})$ encloses the point $\bar{W}$ which lies without $\bar{k}$.

It follows that for every point W within α the circle k_{OW} contains infinitely many points. Let β denote the simple closed curve composed of the arc AXB of the curve α together with that arc of $\bar{k}$ which has A and B as end-points but does not contain P. The interior of β equals the interior of $\bar{k}$ plus the interior of α plus the segment APB of α. It follows with the aid of Theorem 31 that there exists a proper circle k^* with proper interior and with center at O such that $\bar{k}$ is within k^*.

THEOREM 34. *Every circle is a proper circle.*

Proof. Suppose there exists a point O such that not every circle with center at O is a proper circle. Then the set of all points is the sum of two mutually exclusive sets S_1 and S_2 such that every point of S_1 lies on a proper circle with center at O and with proper interior, but no point of S_2 lies on any such circle.

It can easily be seen with the aid of Theorem 33 that S_1 contains no limit point of S_2. Let $\overline{P}$ denote some definite point in S_2. By Theorems 9, 10, 11 and 21 the set of all points is a connected domain. It follows by Theorem 23 that there exists a simple continuous arc $O\overline{P}$ from O to $\overline{P}$. There exists a point X which in the order from O to $\overline{P}$ on $O\overline{P}$ is the first point on $O\overline{P}$ that does not belong to S_1. Let OX denote the interval of OP whose end-points are O and X. If $\overline{Y}$ is a point of S_1 then OX has a point O within, and a point X without, the circle with center at O which passes through $\overline{Y}$. Hence this circle contains a point of OX. It follows that for every point $\overline{Y}$ of S_1 there exists a point Y on OX and a motion M such that $M(O)=O$ and $M(Y)=\overline{Y}$. The point-set S_1 is non-compact, otherwise it would* be bounded and its boundary would be a circle with center at O and passing through X, and this circle would be a simple closed curve enclosing O. Hence, there exists in S_1 an infinite set N of distinct points that has no limit point. There exists a set of motions $M_1, M_2, M_3, \ldots.$, all rotations about O, a point P on OX and a set of points $P_1, P_2, P_3, \ldots.$ all lying in S_1 and on OX such that P is a sequential limit point of the sequence $P_1, P_2, P_3, \ldots.$ and such that $M_1(P_1)$, $M_2(P_2)$, $M_3(P_3), \ldots.$ are distinct points of N. Let k denote a definite circle with center at O and lying in S_1 and let C denote a definite point on k.† There exists a sequence of distinct positive integers $n_1, n_2, n_3, \ldots.$ and a point $\overline{C}$ on k such that $\overline{C}$ is the sequential limit point of the sequence $M_{n_1}(C), M_{n_2}(C), M_{n_3}(C), \ldots.$. There exists a motion M such that $M(O)=O$ and $M(C)=\overline{C}$. There exists a region R containing the arc OX. The sequence $n_1, n_2, n_3, \ldots.$ contains an infinite subsequence of distinct integers $j_1, j_2, j_3, \ldots.$ such that $M_{j_1}(P_{j_1})$, $M_{j_2}(P_{j_2}), M_{j_3}(P_{j_3}), \ldots.$ are all without the region $M(R)$. For each i the arc $M_{j_i}(OP_{j_i})$ contains a point $\overline{E}_{j_i}$ on the boundary of $M(R)$. There exists a point $\overline{E}$ on the boundary of $M(R)$ and an infinite sequence of distinct integers $k_1, k_2, k_3, \ldots.$ of the sequence $j_1, j_2, j_3, \ldots.$ such that $\overline{E}$ is the sequential limit point of $\overline{E}_{k_1}, \overline{E}_{k_2}, \overline{E}_{k_3}, \ldots.$. There exists on OX a sequence of points $E_{k_1}, E_{k_2}, E_{k_3}, \ldots.$, all belonging to S_1, such that, for every i, $\overline{E}_{k_i}=M_{k_i}(E_{k_i})$. There exists a point E on OX and an infinite sequence $m_1, m_2, m_3, \ldots.$ of distinct integers belonging to the sequence $k_1, k_2, k_3, \ldots.$ such that E is the sequential limit point of the sequence $E_{m_1}, E_{m_2}, E_{m_3}, \ldots.$. The point $\overline{C}$ is a sequential limit point of the sequence $M_{m_1}(C), M_{m_2}(C), M_{m_3}(C), \ldots.$, the point O is a sequential

---

*If K is a compact set of points, K' is compact and closed. But every point of K' is in some region. It follows by Theorem 18 and Axiom 5 that K is bounded.

† From here on the present proof bears a certain relationship to an argument of Hilbert's on pp. 407 and 408, *loc. cit.*

limit point of $M_{m_1}(O)$, $M_{m_2}(O)$, $M_{m_3}(O)$, ..., and the point $\overline{E}$ is a sequential limit point of $M_{m_1}(E_{m_1})$, $M_{m_2}(E_{m_2})$, $M_{m_3}(E_{m_3})$, ... It follows that there exists a motion M^* such that $M^*(O)=O$, $M^*(C)=\overline{C}$ and $M^*(E)=\overline{E}$. Since $M^*(O)=M(O)=O$ and $M^*(C)=M(C)$ therefore, by Theorem 32, $M^*(E_{m_1})=M(E_{m_1})$, $M^*(E_{m_2})=M(E_{m_2})$, ... But the point E is a sequential limit point of E_{m_1}, E_{m_2}, E_{m_3}, ..., the point $M(E)$ is a sequential limit point of $M(E_{m_1})$, $M(E_{m_2})$, $M(E_{m_3})$, ..., and $M^*(E)$ is the sequential limit point of $M^*(E_{m_1})$, $M^*(E_{m_2})$, $M^*(E_{m_3})$, ... It follows that $M^*(E)=M(E)$. But $M^*(E)$ is on the boundary of $M(R)$ while $M(E)$ is in $M(R)$. Thus the supposition that there exist points that are not in S_1 has led to a contradiction. It follows that every point distinct from O is on a simple closed curve that encloses O and is a circle with center at O.

THEOREM 35. *There exists an infinite system of circles $k_1, k_2, k_3, \ldots.$ such that for every n, k_n is within k_{n+1} and such that every point is within some k_n.*

Proof. Let O denote some definite point. By Theorem 22 there exists a countably infinite set of distinct points X_1, X_2, X_3, ... such that $X_1+X_2+X_3+\ldots.$ has no limit point. Through the first X_n that is distinct from O there passes a circle k_1 with center at O. Through the first X_n that is without k_1 there is another such circle k_2. This process may be continued. It follows that there exists a sequence $k_1, k_2, k_3, \ldots.$ of circles with center at O such that, for every n, k_{n+1} encloses both k_n and the point-set

$$O+X_1+X_2+\ldots.+X_n.$$

If Z is a point distinct from O, and k is the circle with center at O that contains Z, there exists a positive integer m such that X_m is without k. The circle k_m encloses Z. The truth of Theorem 35 is therefore established.

THEOREM 36. *The set of all points $\overline{S}$ is a number plane in the sense that between $\overline{S}$ and the set of all sensed pairs of real numbers there is a one-to-one correspondence such that in $\overline{S}$ the point $\overline{P}$ is a sequential limit point of the sequence of points $\overline{P}_1, \overline{P}_2, \overline{P}_3, \ldots.$ if and only if $\lim_{n=\infty} x_n=x$ and $\lim_{n=\infty} y_n=y$, where (x, y) is the pair of real numbers corresponding to P and, for every n, (x_n, y_n) is the pair corresponding to $\overline{P}_n$.*

Proof. Let $k_1, k_2, k_3, \ldots.$ be a sequence of circles satisfying the requirements of Theorem 35, and let $R_1, R_2, R_3, \ldots.$ be their interiors. Let both R_{-1} and R_0 denote the null-set. For every positive n let E_n denote the point-set R_n-R_{n-1}. For every n let $K_{E'_n1}, K_{E'_n2}, K_{E'_n3}, \ldots.$ be an infinite sequence

of regions satisfying with respect to the point-set E'_n the requirements of Theorem 19 and such that each of them is a subset of $R_{n+1}-R_{n-2}$. Let $K_1, K_2, K_3, \ldots.$ denote the sequence of regions

$$K_{E'_1 1}, K_{E'_1 2}, K_{E'_2 1}, K_{E'_1 3}, K_{E'_2 2}, K_{E'_3 1}, K_{E'_1 4}, K_{E'_2 3}, K_{E'_3 2}, K_{E'_4 1}, K_{E'_1 5}, \ldots.$$

It is easy to see that the sequence $K_1, K_2, K_3, \ldots.$ satisfies, with respect to $\bar{S}$, the requirements of Axiom 1 of F. A. As has already been shown, if in the statement of Axioms 2–8 of F. A. the term *region* is restricted so as to apply only to Jordan regions, the so modified axioms are all fulfilled in $\bar{S}$. It follows* that Theorem 36 is true.

It having been established that $\bar{S}$ is a number plane, one may now follow Hilbert and arrive at the conclusion that if *straight lines*, *congruence*, etc., are defined as he defines them, the so determined geometry is Euclidean or Bolyai–Lobachevskian, according as our group of motions has or has not an invariant subgroup other than the identity. In this sense *every geometry that satisfies the axioms of Σ is either a Euclidean or a Bolyai–Lobachevskian geometry of two dimensions.*

§ 6. *Independence Examples.*

The symbol E_n will be used to denote an example of a system in which Axiom n is false, but all the other axioms of the set Σ are true. In each example E_n, except E_1, use is made of a well-defined space S_n. In every case the *points* of E_n are the ordinary poins of S_n, but the *regions* of the various E_n's are defined in various ways.

E_1. In E_1 the terms *point, region* and *motion* have no significance.

E_2. S_2 is Euclidean space of two dimensions. A point-set M is a *region* if and only if $M=I$ or $M=I-P$ where I is the interior of some Jordan curve and P is a point in I. A *motion* is an ordinary two-dimensional motion that does not change sense on any closed curve.

E_3. S_3 is Euclidean space of two dimensions. A *region* is the interior of a Jordan curve whose maximum diameter is equal to or greater than 1. *Motion* has the same significance as in E_2.

E_4. S_4 is the linear continuum $(-\infty \leqq x \leqq \infty)$. A *region* is a segment. A *motion* is a one-dimensional rigid motion that does not reverse order.

E_5. S_5 is a point-set composed of a countably infinite set of equal spheres $K_1, K_2, K_3, \ldots.$, all lying in a fixed Euclidean space E of three dimensions, such that every K_n is wholly without every other one. A *motion* is a one-to-

* Cf. proof of Theorem 29.

one transformation of S_n into itself which results from first permuting or leaving fixed the spheres of the set and then rotating each sphere about one of its diameters. A set of points M is a *region* if and only if it is one of the two point-sets into which one of the spheres is separated by a closed Jordan curve lying on it.

E_6. S_6 is Euclidean space of three dimensions. A set of points is a *region* if and only if it is the interior of a cube. Motion has its usual significance.

E_7. S_7 is Euclidean space of two dimensions. A set of points is a *region* if and only if it is the interior of a closed Jordan curve. The only motion is the identity transformation.

E_8. $S_8=S_7$. Motion has the same meaning as in E_2. The interior of every simple closed curve is a *region*. Every *region*, with the exception of a certain *region* R_0, is the interior of a simple closed curve. The *region* R_0 is the domain enclosed by the point-set k_1+k_2 described on page 162 of F. A.

E_9. $S_9=S_7$. *Region* the same as in E_7. Every one-to-one continuous transformation of S_9 into itself that does not change sense is a *motion*.

E_{10}. Same space and same *regions* as in E_7. Select a system of rectangular coordinates. Let $\bar{M}$ denote the transformation of S_{10} into itself represented by the equations $x'=2x$, $y'=2y$. A transformation is a motion if and only if it can be expressed in the form $M_1M_2M_3\ldots M_n$ where $M_1, M_2, \ldots, M_n$ is a finite set of transformations, distinct or otherwise, such that for each $i(1\leqq i\leqq n)M_2$ is either the transformation $\bar{M}$ or some rigid motion that leaves sense invariant.

E_{11}. Same space and same *regions* as in E_{10} Let $\bar{M}$ have the same meaning as in E_{10} and let $\bar{M}^{-1}$ represent the inverse of $\bar{M}$. A one-to-one transformation of S_{11} into itself is a motion if and only if it is identical with $\bar{M}$ or with $\bar{M}^{-1}$, or with some ordinary two-dimensional rigid motion that leaves sense invariant.

E_{12}. Same space and same *regions* as in E_7. A *motion* is a one-to-one continuous transformation of S_{12} into itself that carries lines into lines and preserves distances.

UNIVERSITY OF PENNSYLVANIA.